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Integration on Hyperspaces

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Abstract. In this paper first we present elements of a theory of integration of a quasi-uniform conoid valued functions with respect to a positive measure following the doctoral thesis of the first author "Integration on Uniform Type Conoids". Then we show that it is possible to apply this theory for integration of functions with values in the hyperspace of non-empty convex subsets of a given quasi-uniform conoid. To realize such a possibility we treat the considered hyperspace as a quasi-uniform conoid in a rather natural way (as this is done in our joint work "Uniform type Hyperspaces").

Keywords: Topological conoid, quasi-uniform monoid, quasi-uniform conoid

1 Introduction

The Lebesgue type integration scheme for functions (or measures) with values in different type uniform structures was an object of study by several authors (see [12, 21, 25, 19]). In [1] it was treated for the first time the case of quasi-uniform type structures and it was made an attempt to clarify the role of symmetry for Integration Theory.

A quasi-uniform conoid [1] is a conoid (to be defined) together with a quasi-uniformity such that the conoid addition is uniformly continuous. A uniform version of the corresponding notion was considered earlier in [26] (see also [11, 13, 27]). The class of quasi-uniform conoids includes the class of asymmetric normed

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linear spaces of the recent works [4, 10, 23, 24], where for them the basic questions of functional analysis are investigated.

First, for a quasi-uniform space valued functions we introduce the quasi-uniformity of convergence in measure. Then for a quasi-uniform conoid valued simple functions we define integral and introduce the quasi-uniformity of mean convergence. The class of integrable functions and integrals for them is define in a natural way for quasi-uniform conoids having the Integral Uniqueness Property (IUP).

The second part deals with integration of functions with values in a given hyperspace over a quasi-uniform conoid. Quasi-uniformities on hyperspaces are well-studied (see [5, 15, 16]). According to [2, 3] the quasi-uniform hyperspace of closed convex subsets of a given quasi-uniform conoid inherits its several important properties. We use these facts and show that integrability and integral can be introduced for measurable set-valued functions directly (i.e. without using measurable selections).

2 Integration on Quasi-uniform Conoids

2.1 Quasi-Uniform Spaces

In the following $\mathbb{R}$ will denote the set of real numbers and $\mathbb{R}_+ := [0,\infty[$. The set $\mathbb{R}$ and its subsets (including $\mathbb{R}_+$ and the unit segment $[0,1]$) will be supposed to be endowed with the usual topology $\mathfrak{e}$, and the usual uniformity $\mathcal{E}$.

Let $(X,\mathcal{Q})$ be a quasi-uniform space, and $\mathrm{T}(Q) := \{(y,x) \in X \times X | (x,y) \in Q\}$, when $Q \in \mathcal{Q}$, be the conjugate relation of Q. Instead $\mathrm{T}(Q)$ the notation Q^{-1} is also used. We write $\mathcal{Q}^{\mathrm{T}} = \{\mathrm{T}(Q) \,|\, Q \in \mathcal{Q}\}$. The collection $\mathcal{Q}^{\mathrm{T}}$ is also a quasi-uniformity, called the conjugate quasi-uniformity of $\mathcal{Q}$.

We use usual terminology from the theory of quasi-uniform spaces (see, e.g.,[8], [20],[17]).

A quasi-uniform space $(X,\mathcal{Q})$ is **precompact** if and only if

$\forall Q \in \mathcal{Q}$ there is a finite $F \subset X$ such that $X = Q[F] = \bigcup_{x\in F} Q[x]$.

We say that a quasi-uniformity $\mathcal{Q}$ is:

1. **weakly locally symmetric** at $x \in X$ if for every $Q \in \mathcal{Q}$ there is a symmetric entourage $S \in \mathcal{Q}$ such that $S[x] \subset Q[x]$.
2. **weakly locally symmetric** or **point-symmetric** if $\mathcal{Q}$ is weakly locally symmetric at x for every $x \in X$.

3. **locally symmetric** at $x \in X$ if for every $Q \in \mathcal{Q}$ there is a symmetric entourage $S \in \mathcal{Q}$ such that $S \circ S[x] \subset Q[x]$.

4. **locally symmetric** if $Q \in \mathcal{Q}$ is locally symmetric at x for every $x \in X$.

Let X be a set and $(\mathcal{Q}_i)_{i\in I}$ be a non-empty family of quasi-uniformities in X. Then we denote:

- $\vee_{i\in I}\mathcal{Q}_i$ its least upper bound;
- $\wedge_{i\in I}\mathcal{Q}_i$ its greatest lower bound.

For any family $(\mathcal{Q}_i)_{i\in I}$, $\vee_{i\in I}\mathcal{Q}_i$ and $\wedge_{i\in I}\mathcal{Q}_i$ always exist, moreover $\{\cap_{i\in I_0} Q_i \mid Q_i \in \mathcal{Q}_i, I_0 \text{ is finite}\}$ is a base of $\vee_{i\in I}\mathcal{Q}_i$. For a given quasi-uniformity $\mathcal{Q}$, we denote $\mathcal{Q}^{\vee} = \mathcal{Q} \vee \mathcal{Q}^{\mathrm{T}}$ and $\mathcal{Q}_{\wedge} = \mathcal{Q} \wedge \mathcal{Q}^{\mathrm{T}}$. It is known that $\mathcal{Q}^{\vee}$ is the coarsest **uniformity** containing $\mathcal{Q}$ and $\mathcal{Q}_{\wedge}$ is the finest **uniformity** contained into $\mathcal{Q}$.

For quasi-uniform spaces $(X_1, \mathcal{Q}_1)$ and $(X_2, \mathcal{Q}_2)$ the product $(X_1 \times X_2, \mathcal{Q}_1 \otimes \mathcal{Q}_2)$ is defined in usual way. Let us recall that the product quasi-uniformity $\mathcal{Q}_1 \otimes \mathcal{Q}_2$ has the base $\{\iota(Q' \times Q'') \mid Q' \in \mathcal{Q}_1,\ Q'' \in \mathcal{Q}_2\}$, where

$\iota : (X_1 \times X_1) \times (X_2 \times X_2) \to (X_1 \times X_2) \times (X_1 \times X_2)$

stands for the mapping: $\big((x'_1, x''_1), (x'_2, x''_2)\big) \mapsto \big((x'_1, x'_2), (x''_1, x''_2)\big)$.

2.2 Quasi-uniform Conoids

A semigroup is a pair $(X, +)$, where X is a non-empty set and $+ : X \times X \to X$ is an associative binary operation. A semigroup $(X, +)$ is called Abelian (or commutative) if $x + y = y + x \ \forall x, y \in X$.

An Abelian monoid is a triplet $(X, +, \theta)$, where $(X, +)$ is an Abelian semigroup and $\theta \in X$ an element such that $x + \theta = \theta + x = x$ for every $x \in X$.

If n is a natural number and $x_1, \ldots, x_n$ is a finite sequence of elements of a semigroup X, then the meaning of the expression $\sum_{k=1}^{n} x_k$ is obvious; also it is clear that when X is Abelian, then $\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} x_{\pi(k)}$ for every permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

If $(x_i)_{i\in I}$ is a non-empty family of elements *of an Abelian semigroup* X, then for every *non-empty finite* $J \subset I$ the unordered sum $\sum_{i\in J} x_i$ is defined too; when X is a monoid, it is convenient to set: $\sum_{i\in\varnothing} x_i := \theta$.

If X_1, X_2 are (Abelian) monoids, then the product $X_1 \times X_2$ always is (an Abelian) monoid with respect to the component-wise binary operation and with the neutral element (θ, θ). The same is true for an arbitrary product.

As usual, for non-empty subsets A, B of a semigroup $A + B$ will stand for their algebraic sum: $\{a + b | \ a \in A, b \in B\}$.

An Abelian monoid X which is also a topological space is called a topological Abelian monoid if $+$ is continuous (of course with respect to the product topology in $X \times X$ and the topology of X).

A monoid (semigroup) X equipped with a quasi-uniformity $\mathcal{Q}$ is called **a quasi-uniform monoid (semigroup)** if $+$ is **uniformly continuous** with respect to the product quasi-uniformity $\mathcal{Q} \otimes \mathcal{Q}$ and $\mathcal{Q}$.

Note that a monoid (semigroup) $(X, +)$ equipped with a quasi-uniformity $\mathcal{Q}$ is a quasi-uniform monoid (semigroup) if and only if for every $Q \in \mathcal{Q}$ there is $Q' \in \mathcal{Q}$ such that $Q' + Q' \subset Q$.

We introduce **a conoid** as an Abelian monoid X for which an external operation

$m : X \times \mathbb{R}_+ \to X, m(x, \alpha) = x \cdot \alpha$

is defined with the properties:

- A.1 $(x_1 + x_2) \cdot \alpha = x_1 \cdot \alpha + x_2 \cdot \alpha \quad \forall x_1, x_2 \in X, \quad \forall \alpha \in \mathbb{R}_+$;
- A.2 $(x \cdot \alpha_1) \cdot \alpha_2 = x \cdot (\alpha_1 \cdot \alpha_2) \quad \forall x \in X, \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}_+$;
- A.3 $x \cdot (\alpha_1 + \alpha_2) = x \cdot \alpha_1 + x \cdot \alpha_2 \quad \forall x \in X, \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}_+$;
- A.4 $x \cdot 1 = x \quad \forall x \in X$.

For every $x \in X$ we will consider the mapping:

$$m_x : \mathbb{R}_+ \to X := \alpha \mapsto x \cdot \alpha$$

and for every $\alpha \in \mathbb{R}_+$ we will consider the mapping:

$$m_\alpha : X \to X := x \mapsto x \cdot \alpha.$$

In the literature sometimes a conoid is called an *abstract convex cone* [26], a *cone* [13], a *semi-vector space* [22], or a *semilinear space* [11, 9, 27], etc. The term conoid appeared in [6] and [1].

Let $(X, +, \theta, m)$ be a conoid, K be a non-empty subset of X, $\alpha \in \mathbb{R}_+$ and A non-empty subset of $\mathbb{R}_+$. We denote:

$K \cdot \alpha = \{x \cdot \alpha | \, x \in K\}$ and $K \cdot A = \{x \cdot \alpha | \, x \in K, \; \alpha \in A\}$.

For a non-empty subset Q of $X \times X$, we put

$Q \cdot \alpha = \{(x \cdot \alpha, y \cdot \alpha) | \, (x, y) \in Q\}$ and $Q \cdot A = \{(x \cdot \alpha, y \cdot \alpha) | (x, y) \in Q, \; \alpha \in A\}$.

Let $(X, +, \theta, m)$ be a conoid. A subset K of X is called:

1. **Convex** if either K is empty, or $K \cdot \alpha + K \cdot (1 - \alpha) \subset K$, for every $\alpha \in [0,1]$.
2. **Balanced** if either K is empty, or $K \cdot [0,1] \subset K$.

In the usual way, we will denote $co(K)$ the convex hull of a subset $K \subset X$.

A conoid X equipped with a topology τ will be called **a topological conoid** if $(X, +, \theta, \tau)$ is a topological monoid and the mapping

$m : X \times \mathbb{R}_+ \to X, m(x, \alpha) = x \cdot \alpha$

is continuous with respect to topologies $\mathfrak{e} \otimes \tau$ and τ.

A conoid X equipped with a topology τ will be called **Mtopological conoid** if $(X, +, \theta, \tau)$ is a topological monoid.

Therefore a Mtopological conoid is simply a topological monoid which algebraically is a conoid.

We say that a Mtopological conoid $(X, +, \theta, m, \tau)$ *has the property* (C_{cont}) if $m_x : \mathbb{R}_+ \to X$ is $(\mathfrak{e}, \tau)-$continuous on $\mathbb{R}_+$ for every $x \in X$.

A conoid X equipped with a quasi-uniformity $\mathcal{Q}$ will be called **a quasi-uniform conoid** if $(X, +, \theta, \mathcal{Q})$ is a quasi-uniform monoid and the mapping

$$m : X \times \mathbb{R}_+ \to X, m(x, \alpha) = x \cdot \alpha$$

is continuous with respect to topologies $\mathfrak{e} \otimes \tau_{\mathcal{Q}}$ and $\tau_{\mathcal{Q}}$.

A conoid X equipped with a quasi-uniformity $\mathcal{Q}$ will be called **a Mquasi-uniform conoid** if $(X, +, \theta, \mathcal{Q})$ is a quasi-uniform monoid.

Therefore a Mquasi-uniform conoid is simply a uniform monoid which algebraically is a conoid.

We say that a Mquasi-uniform conoid $(X, +, \theta, m, \mathcal{Q})$ *has the property* (C_{cont}) if $m_x : \mathbb{R}_+ \to X$ is continuous with respect to topologies $\mathfrak{e}$ and $\tau_{\mathcal{Q}}$ on $\mathbb{R}_+$ for every $x \in X$.

We say that a Mquasi-uniform conoid $(X, +, \theta, m, \mathcal{Q})$ *has the property* $(C_{\text{cont}}^{\mathrm{T}})$ if $m_x : \mathbb{R}_+ \to X$ is continuous with respect to topologies $\mathfrak{e}$ and $\tau_{\mathcal{Q}^{\mathrm{T}}}$ on $\mathbb{R}_+$ for every $x \in X$.

We shall say that a Mquasi-uniform conoid $(X,+,\theta,m,\mathcal{Q})$ is

- **locally convex** if $\mathcal{Q}$ admits a base consisting of convex entourages;
- **locally balanced** if $\mathcal{Q}$ admits a base consisting of balanced entourages.

Clearly, a locally convex quasi-uniform conoid is locally balanced too.

2.3 Submeasures, Premeasures and Measures

In this subsection we introduce the notations and essential properties to develop a integration scheme.

Let Ω be a non-empty set. A collection of subsets of Ω is called:

--- *a ring* if

$\mathcal{A} \neq \varnothing, A, B \in \mathcal{A} \quad \Rightarrow \quad A \cup B \in \mathcal{A},\ A \setminus B \in \mathcal{A}$

--- *an algebra* if $\mathcal{A}$ is a ring such that $\Omega \in \mathcal{A}$.

--- *a* σ *-ring* if $\mathcal{A}$ is a ring such that $A_n \in \mathcal{A}, n = 1,2,\ldots \Rightarrow \cup_{n=1}^{\infty} A_n \in \mathcal{A}$.

--- *a* σ *- algebra* if $\mathcal{A}$ is a σ -ring such that $\Omega \in \mathcal{A}$.

In what follows $\mathcal{A}$ *will always stand for an algebra of subsets of* Ω.

If $(X,+)$ is a semigroup, then a set function $\varphi : \mathcal{A} \to X$ with property $A, B \in \mathcal{A}, A \cap B = \varnothing \Rightarrow \varphi(A \cup B) = \varphi(A) + \varphi(B)$, is called *additive*.

If $(X,+,\theta)$ is a monoid, then an additive set function $\varphi : \mathcal{A} \to X$ with property $\mu(\varnothing) = \theta$ will be called *premeasure.*

If $(X,+,\tau)$ is a topological semigroup, then a set function $\varphi : \mathcal{A} \to X$ with property

$A_1, A_2, \ldots, A_n, \ldots, A \in \mathcal{A}, A_k \cap A_j = \varnothing$ when $k \neq j$ and

$$\cup_{k=1}^{\infty} A_k = A \Rightarrow \varphi(A) = \lim_n \sum_{k=1}^{n} \varphi(A_k)$$

is called *countably additive* or σ -additive.

If $(X,+,\theta,\tau)$ is a topological monoid, then a countably additive set function $\varphi : \mathcal{A} \to X$ with property $\varphi(\varnothing) = \theta$ will be called *a* X*-valued measure*, or simply, *a measure* when the range will be clear from the context.

Since $\mathbb{R}_+$ and $\overline{\mathbb{R}_+} := [0,\infty]$ are topological monoids with respect to the usual addition and topologies, the above given definitions can be applied for them. When we speak about a non-negative set function, we mean a $\overline{\mathbb{R}_+}$ -valued set function.

A set function $\mu:\mathcal{A}\to 0,\infty]$ is called:

--- *increasing*, if $A,B\in\mathcal{A},\ A\subset B\Rightarrow \mu(A)\le\mu(B)$.

--- *subadditive* , if $A,B\in\mathcal{A}\Rightarrow \mu(A\cup B)\le\mu(A)+\mu(B)$;

--- *countably subadditive* , if

$A_1,A_2,\ldots,A_n,\ldots,A\in\mathcal{A}$, and

$$\cup_{k=1}^{\infty} A_k = A\Rightarrow \mu(A)\le\lim_n\sum_{k=1}^{n}\mu(A_k).$$

Any subadditive and increasing set function $\mu:\mathcal{A}\to[0,\infty],\ \mu(\varnothing)=0$ is called *a submeasure.*

Clearly, any non-negative premeasure is a submeasure as well.

To a given set function $\nu:\mathcal{A}\to 0,\infty]$ we associate *the outer set function*

$\nu^e:\mathcal{P}(\Omega)\to[0,\infty]$ by the equality: $\nu^e(C)=\inf\{\nu(A)|C\subset A \text{and} A\in\mathcal{A}\}$,

where we agree that $\inf(\varnothing)=\infty\in[0,\infty]$.

Clearly, the restriction of ν^e to $\mathcal{A}$ coincides with ν; moreover,

--- ν^e is increasing,

--- if ν is subadditive, then ν^e is a subadditive, i.e. it is a submeasure.

--- if ν is countably subadditive *and $\mathcal{A}$ is a σ-ring, then ν^e is countably subadditive.*

For a given premeasure μ the set function $\mu^e:\mathcal{P}(\Omega)\to[0,\infty]$ will be called *the outer submeasure associated with* μ .

2.4 Simple Functions and their Integral

In what follows a triplet $(\Omega,\mathcal{A},\mu)$, where Ω is a nonempty set, $\mathcal{A}$ is an *algebra* of subsets of Ω and $\mu:\mathcal{A}\to\mathbb{R}_+$ is a premeasure will be called *a finite premeasure space*.

A triplet $(\Omega,\mathcal{A},\mu)$, where Ω is a nonempty set, $\mathcal{A}$ is a σ*-algebra* of subsets of Ω and $\mu:\mathcal{A}\to\mathbb{R}_+$ is a premeasure will be called *a finite measure space*.

In this subsection $(\Omega,\mathcal{A},\mu)$ will be a finite premeasure space and X will be a conoid (the assumption that μ is finite simplifies presentation; this assumption is not made in [1]).

A mapping $s:\Omega\to X$ will be called $\mathcal{A}-$ **simple** if $s(\Omega)$ is finite and $s^{-1}(\{x\})\in\mathcal{A},\ \forall x\in s(\Omega)$.

The set of all $\mathcal{A}$ - simple functions will be denoted $\mathcal{S}(\mathcal{A};X)$ or, briefly, $\mathcal{S}$.

Observe that $\mathcal{S}$ is a conoid with respect to usual addition and multiplication on non-negative real numbers.

For a $\mathcal{A}$ – simple function s and for $M \in \mathcal{A}$ we write:

$$\int_M s d\mu := \sum_{x \in s(\Omega)} x \cdot \mu\left(M \cap s^{-1}(\{x\})\right).$$

The following properties can be verified easily:

$$\int_M (s_1 \cdot \alpha_1 + s_2 \cdot \alpha_2) d\mu =$$

$$\left(\int_M s_1 d\mu\right) \cdot \alpha_1 + \left(\int_M s_2 d\mu\right) \cdot \alpha_2, \forall s_1, s_2 \in \mathcal{S}, \forall M \in \mathcal{A}, \forall \alpha_1, \alpha_2 \in \mathbb{R}_+,$$

$$M_1, M_2 \in \mathcal{A}, M_1 \cap M_2 = \varnothing \Rightarrow \int_{M_1 \cup M_2} s d\mu = \int_{M_1} s d\mu + \int_{M_2} s d\mu, \forall s \in \mathcal{S}.$$

The **indefinite integral** for a $\mathcal{A}$ – simple function $s : \Omega \to X$ is defined by:

$$\mathcal{I}_s : \mathcal{A} \to X \quad \text{such that} \quad M \mapsto \int_M s\, d\mu.$$

For every $s \in \mathcal{S}$ the indefinite integral $\mathcal{I}_s : \mathcal{A} \to X$ is a X-valued premeasure.

It is easy to see that if μ is *a measure* and X is a Mtopological conoid *with property* (C_{cont}), then for every $s \in \mathcal{S}$ the indefinite integral $\mathcal{I}_s : \mathcal{A} \to X$ is a X-valued *measure*.

In Subsection 2.6 we extend the notion of integral for a wider class of consider quasi-uniform conoid valued functions.

2.5 Convergence in Measure and Uniform Convergence

2.5.1 Quasi-Uniformities of uniform and pointwise convergences

Let:

--- E, X be non-empty sets,

--- X^E be the set of all mappings $\phi : E \to X$ and Φ is a non-empty subset of X^E;

--- $Q \subset X \times X$ and $K \subset E$,

--- $Q_K(\Phi) = \{(\phi, \psi) \in \Phi \times \Phi | (\phi(e), \psi(e)) \in Q \quad \forall e \in K\}$ and $Q_K := Q_K(X^E)$.

--- $\mathcal{Q}$ is a quasi-uniformity in X,

--- $\mathbb{B}_K(\Phi) = \{Q_K(\Phi) | Q \in \mathcal{Q}\}$,

--- $\mathfrak{B}_K(\Phi) = \{Q_{\{e\}}(\Phi) | Q \in \mathcal{Q}, e \in K\}$.

The quasi-uniformity in Φ having the base $\mathbb{B}_K(\Phi)$, which we denote $\mathbb{Q}_K(\Phi)$ or $\mathbb{U}_K(\mathcal{Q};\Phi)$, is called *the quasi-uniformity of uniform convergence* on K (associated with $\mathcal{Q}$).

When $\Phi = X^E$ we use shorter notations: $\mathbb{Q}_K := \mathbb{Q}_K(X^E)$ or $\mathbb{U}_K(\mathcal{Q}) := \mathbb{U}_K(\mathcal{Q}; X^E)$.

The topology induced by $\mathbb{Q}_K(\Phi)$ in Φ is called *the topology of uniform convergence on* K; if a net $(\phi_i)_{i\in I} \subset \Phi$ converges to a $\phi \in \Phi$ in $(\Phi, \mathbb{Q}_K(\Phi))$, then we say that $(\phi_i)_{i\in I}$ *converges to* ϕ *uniformly on* K.

The quasi-uniformity in Φ having the base $\mathfrak{B}_K(\Phi)$, which we denote $\mathfrak{Q}_K(\Phi)$ or $\mathbb{P}_K(\mathcal{Q};\Phi)$, is called *the quasi-uniformity of pointwise convergence* on K (associated with $\mathcal{Q}$).

When $\Phi = X^E$ we use shorter notations: $\mathfrak{Q}_K := \mathfrak{Q}_K(X^E)$ or $\mathbb{P}_K(\mathcal{Q}) := \mathbb{P}_K(\mathcal{Q}; X^E)$.

The topology induced by $\mathfrak{Q}_K(\Phi)$ in Φ coincides with the topology of pointwise convergence on K: a net $(\phi_i)_{i\in I} \subset \Phi$ converges to a $\phi \in \Phi$ in

$\left(\Phi,\mathfrak{Q}_K(\Phi)\right)$ iff for each $e\in K$ the net $(\varphi_i(e))_{i\in I}\subset X$ converges to $\varphi(e)$ in $(X,\mathcal{T}_Q)$.

We have the inclusion: $\mathfrak{Q}_E\subset\mathbb{Q}_E$, which in many interesting cases is strict.

2.5.2 Quasi-Uniformity of Convergence in Measure

We fix a non-empty set Ω, a subset $\Omega_0\subset\Omega$, a submeasure $\nu:\mathcal{P}(\Omega)\to[0,\infty]$ and a quasi-uniform space $(X,\mathcal{Q})$.

For $f,g\in X^{\Omega}$ and $Q\subset X\times X$ we put:

$[f,g,Q]=\{\omega\in\Omega\,|\,(f(\omega),g(\omega))\not\in Q\}$.

In the set X^{Ω} we introduce now the quasi-uniformity $\mathbb{Q}_\nu$ of *convergence in submeasure* ν. To define it we need one more notation:

$Q_\nu^{\varepsilon}(\Omega_0)=\{(f,g)\in X^{\Omega}\times X^{\Omega}|\nu([f,g,Q]\cap\Omega_0)<\varepsilon\},\forall\varepsilon\in]0,\infty[,\forall Q\subset X\times X.$

We denote $\mathbb{Q}_\nu(\Omega_0)$ the quasi-uniformity generated by the following base:

$\mathbb{B}_\mu(\Omega_0)=\{Q_\mu^{\varepsilon}(\Omega_0)|Q\in\mathcal{Q},\varepsilon\in]0,\infty[\}$

and call it *the quasi-uniformity of convergence in submeasure* ν *on* Ω_0.

If $\Omega_0=\Omega$ then instead of $\mathbb{Q}_\nu(\Omega_0)$ we use a shorter notation $\mathbb{Q}_\nu$ and call this quasi-uniformity *the quasi-uniformity of convergence in submeasure* ν.

If $\mathcal{Q}$ is a uniformity, then $\mathbb{Q}_\nu(\Omega_0)$ also is.

A net convergent in $(X^{\Omega},\mathbb{Q}_\nu(\Omega_0))$ will be called *convergent in submeasure* ν *on* Ω_0

A net convergent in $(X^{\Omega},\mathbb{Q}_\nu)$ will be called *convergent in submeasure* ν.

We compare now $\mathbb{Q}_\nu$ with the quasi-uniformity $\mathbb{Q}_\Omega$ of uniform convergence in X^{Ω} associated with $\mathcal{Q}$.

Lemma 2.1 *Let* $\nu:\mathcal{P}(\Omega)\to\left[0,\infty\right]$ *be a submeasure and* $(X,\mathcal{Q})$ *be a quasi-uniform space. We have:*

(1) Always $\mathbb{Q}_\nu\subset\mathbb{Q}_\Omega$; i.e., the uniform convergence implies the convergence in submeasure ν.

(2) If ν is the counting measure, then $\mathbb{Q}_\nu=\mathbb{Q}_\Omega$.

Proof. (1) is true because $Q_{\Omega} \subset Q_{\nu}^{\varepsilon}, \forall Q \in \mathcal{Q}, \varepsilon \in]0,\infty[$.

(2) is true because $Q_{\Omega} = Q_{\nu}^{\varepsilon}, \forall Q \in \mathcal{Q}, \varepsilon \in]0,1[$. ü

Proposition 2.2

Let $(X,\mathcal{Q})$ and $(X',\mathcal{Q}')$ be quasi-uniform spaces, $\{f_i : \Omega \to X | i \in I\}$ be a net which converges to a $f : \Omega \to X$ in submeasure ν and $u : X \to X'$ be *a uniformly continuous mapping*.

Then the net $\{u \circ f_i : \Omega \to X' | i \in I\}$ converges to $u \circ f : \Omega \to X'$ in submeasure ν.

Proof. We need show that given $Q' \in \mathcal{Q}'$ and $\varepsilon > 0$ there exists $i_0 \in I$ such that $u \circ f_i \in Q_{\nu}'^{\varepsilon}[u \circ f] \quad \forall i \succ i_0$:

By the uniform continuity of u , there exists $Q \in \mathcal{Q}$ such that $(u \times u)(Q) \subset Q'$.

By the convergence in measure of $(f_i)_{i\in I}$ to f given Q and $\varepsilon > 0$ exists $i_0 \in I$ such that $f_i \in Q_{\mu}^{\varepsilon}[f] \quad \forall i \succ i_0$.

Since $[u \circ f, u \circ f_i, Q'] \subset [f, f_i, Q]$ it is clear that $\nu([u \circ f, u \circ f_i, Q']) \le \nu([f, f_i, Q])$ and then $\nu([u \circ f, u \circ f_i, Q']) < \varepsilon \quad \forall i \succ i_0$.

Let $\nu = \mu^e$ for a premeasure (measure) μ given on an algebra $\mathcal{A}$. Then:

--- $\mathbb{Q}_{\mu^e}(\Omega_0)$ will be called the quasi-uniformity of *convergence in premeasure (measure)* μ on Ω_0.

--- we will use simplified notations : $\mathbb{Q}_{\mu}(\Omega_0) := \mathbb{Q}_{\mu^e}(\Omega_0)$ and $\mathbb{Q}_{\mu} := \mathbb{Q}_{\mu^e}$.

The closure of $\mathcal{S}$ into $(X^{\Omega}, \mathcal{Q}_{\mu})$ will be denoted by $L_0(\mu; X)$ and the members of $L_0(\mu; X)$ will be called μ - **measurables.**

2.6 Integrable Functions and Their Integral

In this subsection $(\Omega, \mathcal{A}, \mu)$ will be a finite premeasure space and $(X,\mathcal{Q})$ will be a Mquasi-uniform conoid such that $\tau_{\mathcal{Q}}$ *is a* T_2*-topology*.

We intend to examine the largest class of X-valued functions for which is still possible to define the integral with preservation of ordinary properties.

We say that a function $f : \Omega \to X$ is $\mu-$ **preintegrable**, if there exists a net $(s_i)_{i\in I}$ of $\mathcal{A}-$ simple functions such that

(I) The net $(s_i)_{i\in I}$ converges to f in premeasure μ.

(II). The net $(\mathcal{I}_{s_i})_{i\in I}$ converges in $(X^{\mathcal{A}}, \mathbb{U}_{\mathcal{A}}(\mathcal{Q}))$.

The set of all μ-preintegrable functions $f:\Omega\to X$ is denoted by **PrInt** $(\mu;X)$.

It is clear that $\mathcal{S}\subset$ **PrInt** $(\mu;X)\subset L_0(\mu;X)$.

For a fixed $f\in \mathrm{PrInt}(\mu;X)$ every net $(s_i)_{i\in I}$ of $\mathcal{A}-$ simple functions satisfying (I) and (II) will be called *a defining net* for f.

To a given $f\in \mathrm{PrInt}(\mu;X)$ let us associate the set $\Phi_f\subset X^{\mathcal{A}}$ as follows:

$\Phi_f := \{\varphi\in X^{\mathcal{A}} | \varphi = \lim_{i\in I}\mathcal{I}_{s_i} \text{ where} (s_i)_{i\in I} \text{is some defining net for} f\}.$

Clearly, $\Phi_f \neq \varnothing$. Observe that if τ_Q *is not a* T_2*-topology* then for each $f\in \mathrm{PrInt}(\mu;X)$ the set Φ_f *contains more than one element*. To exclude such a possibility, from now on **we assume that** τ_Q **is a** T_2**-topology**.

We say that a $f:\Omega\to X$ is μ - **integrable**, if f is μ-preintegrable and the set Φ_f *consists of a unique element*. The set of all μ-integrable functions $f:\Omega\to X$ we denote by **Int** $(\mu;X)$. Let $f\in \mathrm{Int}(\mu;X)$; then the set Φ_f consists of the unique set function $\varphi_f:\mathcal{A}\to X$ and we write:

$\int_M fd\mu := \varphi_f(M), M\in\mathcal{A}.$

In other words, we set by definition:

$\int_M f\,d\mu := \lim_{i\in I}\int_M s_i\,d\mu \quad \forall M\in\mathcal{A},$

where $(s_i)_{i\in I}$ is *some defining net for* f.

The introduced integral has the expected properties:

$$\int_M (f_1\cdot\alpha_1 + f_2\cdot\alpha_2)d\mu =$$

$$\left(\int_M s_1 d\mu\right)\cdot\alpha_1 + \left(\int_M s_2 d\mu\right)\cdot\alpha_2, \forall f_1, f_2\in Int(\mu;X), \forall M\in\mathcal{A}, \forall\alpha_1,\alpha_2\in\mathbb{R}_+,$$

$$M_1, M_2\in\mathcal{A}, M_1\cap M_2 = \varnothing \Rightarrow \int_{M_1\cup M_2} fd\mu = \int_{M_1} sd\mu + \int_{M_2} fd\mu, \forall f\in Int(\mu;X).$$

For a given $f\in \mathrm{Int}(\mu;X)$ the set function $I_f := \varphi_f$ which maps $\mathcal{A}$ into X will be called *the indefinite integral of* f with respect to μ.

Theorem 2.3 *[1, Proposition 5.4.6] Let* $(X,+,\theta,\mathcal{Q})$ *be a locally symmetric Mquasi-uniform conoid with property* (C_{cont}) *and* μ *be a measure. Then for every* $f \in Int(\mu;X)$ *the indefinite integral* $\mathcal{I}_f : \mathcal{A} \to X$ *is a* X*-valued measure as well.*

Let us underline that the local symmetry is essential for the validity of the last theorem.

In this way we have defined the notion of the integral for all functions from $Int(\mu;X)$. The following statement shows that the class $Int(\mu;X)$ is larger than the class of $\mathcal{A}$-simple functions, $\mathcal{S}(\mathcal{A};X)$.

Proposition 2.4 *[1, Proposition 5.4.14] Let* $(X,+,\theta,\mathcal{Q})$ *be a locally symmetric, locally convex Mquasi-uniform conoid with pro-perties* (C_{cont}) *and* $(C^{\mathrm{T}}_{\mathrm{cont}})$. *Then for any measure* μ *on* $\mathcal{A}$ *we have* $\mathcal{S}(\mathcal{A};X) \subset Int(\mu;X)$.

Let us underline also that our exposition does not use any notion of completeness for the considered quasi-uniform spaces, which is crucial for the uniform versions of the Integration theory (cf. [12, 21, 25, 19]).

3 Integration on Uniform Type Hyperspaces

3.1 Quasi-Uniform Hyperspaces

Let $(X,\mathcal{Q})$ be a quasi-uniform space, and let $\mathcal{P}_0(X)$ be the collection of all nonempty subsets of X. We will consider several a quasi-uniformities in $\mathcal{P}_0(X)$ associated with $\mathcal{Q}$. For each $Q \in \mathcal{Q}$, set

$$Q^+ = \{(A,B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) \mid B \subset Q[A]\}$$

and

$$Q^- = \{(A,B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) \mid A \subset \mathrm{T}(Q)[B]\}.$$

Then

$$\mathcal{Q}^+ = \{Q^+ \mid Q \in \mathcal{Q}\}$$

is a base of a quasi-uniformity on $\mathcal{P}_0(X)$ which is called *the upper quasi-uniformity*.

$$\mathcal{Q}^- = \{Q^- \mid Q \in \mathcal{Q}\}$$

is a base of a quasi-uniformity on $\mathcal{P}_0(X)$ which is called *the lower quasi-uniformity*.

The quasi-uniformity $\mathcal{Q}^* = \mathcal{Q}^+ \vee \mathcal{Q}^-$ is called *the Bourbaki quasi-uniformity* (see [5], [18]).

Proposition 3.1 *[cf. [3]]Let* $(X,\mathcal{Q})$ *be a locally symmetric quasi-uniform space; then*

1. $\mathcal{Q}^-$ is locally symmetric at $\{x\}$, $\forall x \in X$.
2. $\mathcal{Q}^+$ is locally symmetric at $\{x\}$, $\forall x \in X$.
3. $\mathcal{Q}^*$ is locally symmetric at $\{x\}$, $\forall x \in X$.

3.2 Quasi-Uniform Hyper-Conoids

If $(X,+,\theta)$ is a monoid, the internal operation can be extended to $\mathcal{P}_0(X)$ in a natural manner

$A+B=\{a+b \mid a\in A, b\in B\}$.

Let $(X,+,\theta,m)$ be a conoid, $\mathcal{P}_0(X)$ be $\mathcal{P}(X)\setminus\{\varnothing\}$. The external operation

$m: X\times\mathbb{R}_+ \to X, m(x,\alpha)=x\cdot\alpha$

can be extended to $\mathcal{P}_0(X)$ in a natural manner:

$m:\mathcal{P}_0(X)\times\mathbb{R}_+\to\mathcal{P}_0(X), m(K,\alpha)=K\cdot\alpha.$

The structure $\left(\mathcal{P}_0(X),+,\{\theta\},m\right)$ may not be a conoid, because, in general,

$K\cdot(\alpha_1+\alpha_2)\neq K\cdot\alpha_1+K\cdot\alpha_2.$

Let $\mathcal{P}_{conv}(X)$ be the collection of all non-empty convex subsets of the conoid X. Then the structure $\left(\mathcal{P}_{conv}(X),+,\{\theta\},m\right)$ *is a conoid.* This is an important example of a conoid. Observe that since $X+X=X$, the conoid $\mathcal{P}_{conv}(X)$ is not cancellative provided $X\neq\{\theta\}$.

Since $\left(\mathcal{P}_{conv}(X),+,\{\theta\},m\right)$ is a conoid, all definitions and observations from Subsection 2.4 are applicable for $\mathcal{P}_{conv}(X)$-valued functions. In particular, if $(\Omega,\mathcal{A},\mu)$ is a finite premeasure space, then for the members of $\mathcal{S}(\mathcal{A};\mathcal{P}_{conv}(X))$ the integral is well-defined and has obvious properties.

In [3] we can find the following results.

Theorem 3.2 *Let* $(X,+,\theta,\mathcal{Q})$ *be a quasi-uniform monoid. Then* $(\mathcal{P}_0(X),+,\mathcal{Q}^-)$, $(\mathcal{P}_0(X),+,\mathcal{Q}^+)$ *and* $(\mathcal{P}_0(X),+,\mathcal{Q}^*)$ *are quasi-uniform monoids.*

Corollary 3.3 *If* $(X,+,m,\mathcal{Q})$ *is a quasi-uniform conoid. Then* $(\mathcal{P}_{conv}(X),+,m,\mathcal{Q}^-)$, $(\mathcal{P}_{conv}(X),+,m,\mathcal{Q}^+)$ *and* $(\mathcal{P}_{conv}(X),+,\{\theta\},m,\mathcal{Q}^*)$ *are also quasi-uniform conoids.*

The definitions and conclusions of Subsection 2.6 are not directly applicable for the quasi-uniform conoids $(\mathcal{P}_{conv}(X),+,m,\mathcal{Q}^{+})$ and $(\mathcal{P}_{conv}(X),+,\{\theta\},m,\mathcal{Q}^{*})$ because the topologies induced by these quasi-uniformities satisfy the T_2-axiom only in trivial cases.

Let now $(X,\mathcal{Q})$ be a T_2-quasi-uniform conoid; denote by $\mathcal{P}_{\mathrm{cl-conv}}(X)$ the collection of all non-empty convex *closed* subsets of X. In many cases $\left(\mathcal{P}_{\mathrm{cl-conv}}(X),\mathcal{Q}^{*}\right)$ is a T_2-quasi-uniform space. However, in general $\mathcal{P}_{\mathrm{cl-conv}}(X)$ is not a subconoid of $\left(\mathcal{P}_{conv}(X),+\right)$ (because it may happen that for some $A,B\in\mathcal{P}_{\mathrm{cl-conv}}(X)$ the set $A+B$ *is not closed*).

Let us denote now by $\mathcal{P}_{\mathrm{comp-conv}}(X)$ the collection of all non-empty convex *compact* subsets of X. In many cases $\left(\mathcal{P}_{\mathrm{cl-conv}}(X),\mathcal{Q}^{*}\right)$ is a T_2-quasi-uniform space. We have also, that $P_{comp-conv}(X)$ *is a subconoid of* $\left(\mathcal{P}_{conv}(X),+\right)$ (because $A,B\in P_{\mathrm{comp-conv}}(X)\Rightarrow A+B\in\mathcal{P}_{\mathrm{comp-conv}}(X)$). Consequently, $\left(\mathcal{P}_{\mathrm{comp-conv}}(X),\mathcal{Q}^{*}\right)$ is always a quasi-uniform conoid and the definitions and conclusions of Subsection 2.6 are directly applicable for $\mathcal{P}_{\mathrm{comp-conv}}(X)$-valued functions. In particular, if $(\Omega,\mathcal{A},\mu)$ is a finite premeasure space, then the class $Int\left(\mu;\mathcal{P}_{\mathrm{comp-conv}}(X)\right)$ of μ-integrable $\mathcal{P}_{\mathrm{comp-conv}}(X)$-valued functions is well-defined and for every $f\in Int\left(\mu;\mathcal{P}_{\mathrm{comp-conv}}(X)\right)$ and $M\in\mathcal{A}$ the integral $\int_M f d\mu$ is a well-defined compact convex subset of X.

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