ISSN: 1645-9911

# Uniform type structures

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**Abstract.** In a non-empty set $X$ a filter $\mathcal{Q}$ on $X \times X$ consisting of reflexive relations, which satisfies:

$$\forall x \in X \ \forall \mathrm{Q} \in \mathcal{Q}, \exists \mathrm{P} \in \mathcal{Q} \text{ such that } \mathrm{P} \circ \mathrm{P}[x] \subset \mathrm{Q}[x]$$

is called local quasi-uniformity. If we require the symmetry condition then we obtain a local uniformity. These concepts were introduced by Fletcher-Lindegren(1974) and Williams(1972) respectively.

We will discuss the possibility of extending the known results about (quasi)-uniform spaces to local (quasi)-uniform spaces

**Keywords:** quasi-uniformity, local quasi-uniformity, local quasi-uniform space, quasi-uniform space, uniform space.

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## 1. Introduction

If $X$ is a non-empty set a filter $\mathcal{Q}$ on $X \times X$ of reflexive relations, which satisfies the divivisiblity condition :

$$\forall \mathrm{Q} \in \mathcal{Q}, \exists \mathrm{P} \in \mathcal{Q} \text{ such that } \top(\mathrm{P}) \circ \mathrm{P} \subset \mathrm{Q}$$

is called uniformity in $X$.

The notion of uniform space was introduced by Weil (1937) as a generalization of the concept of a metric space (see 3.1).

Removing the symmetry Nachbin (1948a), Nachbin (1948b), and Nachbin (1965) obtained the concept of semi-uniform structures. Later, the term quasi-uniformity suggested by Császár (1960) was commonly accepted.

Williams (1972) and Fletcher-Lindgren (1974) introduced the notion of local uniformity and local quasi-uniformity, respectively, localizing the divisibility condition:

$$\forall x \in X \ \ \forall \mathrm{Q} \in \mathcal{Q} \ \ \exists \mathrm{P} \in \mathcal{Q} \text{ such that } \mathrm{P} \circ \mathrm{P}[x] \subset \mathrm{Q}[x].$$

If $\mathcal{Q}$ is a quasi-uniformity, it is known that the conjugate filter $\mathcal{Q}^{\top}$ is always a quasi-uniformity. In this paper we give an example of a local quasi-uniformity $\mathcal{Q}$ such that conjugate filter $\mathcal{Q}^{\top}$ is not a local quasi-uniformity and other example of a local quasi-uniformity which is not quasi-uniformity and the conjugate filter $\mathcal{Q}^{\top}$ is a local quasi-uniformity. In this case we say that $\mathcal{Q}$ is bilocal quasi-uniformity.

The set all local quasi-uniformities in $X$, $\mathrm{LQ}(X)$, with respect to the set-theoretic inclusion $\subset$ is a partially ordered set, with one smallest element, the indiscrete uniformity $\{X \times X\}$ and one biggest element, the discrete uniformity $\{R \in P(X \times X) | R \supset \Delta\}$, hence $(\mathrm{LQ}(X), \subset)$ is a complete lattice.

For a given family of local quasi-uniformities $(\mathcal{Q}_i)_{i \in I}$, we study the local quasi-uniformities $\inf_{i \in I} \mathcal{Q}_i$ and $\sup_{i \in I} \mathcal{Q}_i$. In particular if $\mathcal{Q}$ is bilocal quasi-uniformity, we show that $\inf\left\{\mathcal{Q}, \mathcal{Q}^{\top}\right\}$ and $\sup\left\{\mathcal{Q}, \mathcal{Q}^{\top}\right\}$ are local uniformities.

We presented also, some topological aspects of local quasi-uniform spaces.

## 2. Introductory concepts

In all this paper $X$ be a non-empty set.

For $X$ we define **filter**, $\mathbb{F}$, as a non empty set family of subsets of $X$ such that:

- ✓ $\varnothing \notin \mathbb{F}$.
- ✓ If $A, B \in \mathbb{F} \Rightarrow A \cap B \in \mathbb{F}$.
- ✓ If $A \in \mathbb{F}$ and $A \subset B \Rightarrow B \in \mathbb{F}$.

We define **filter base**, $\mathbb{B}$ as a non empty set family of subsets of $X$ provided:

- ✓ $\varnothing \notin \mathbb{B}$.
- ✓ If $A, B \in \mathbb{B} \Rightarrow \exists C \in \mathbb{B}$ such that $C \subset A \cap B$ .

We can observe that, the whole filter is filter base and a filter base generates the filter

$$F(\mathbb{B}) = \{A \subset X : \exists B \in \mathbb{B} \text{ such that } B \subset A\}.$$

**Remark 2.1**

1. Let $(X, \tau)$ be a topological space, and $\mathcal{N}_\tau(x)$ the family of all neighbourhoods of $x \in X$ , then $\left(\mathcal{N}_\tau(x)\right)_{x \in X}$ has the following properties:
   - ✓ $\mathcal{N}_\tau(x)$ is a filter, $\forall x \in X$.
   - ✓ $x \in X$, $E \in \mathcal{N}_\tau(x)$ implies $x \in \mathcal{N}_\tau(x)$ .
   - ✓ $x \in X, E \in \mathcal{N}_\tau(x)$ implies that $\exists F \in \mathcal{N}_\tau(x)$ such that $\forall y \in F$, $E \in \mathcal{N}_\tau(y)$.

2. The following converse of (1) is true:
   If $X$ is a non-empty set and with every $x \in X$ associated a family $\mathcal{N}(x)$ of subsets of $X$ , with the following properties:
   - ✓ $\mathcal{N}(x)$ is a filter, $\forall x \in X$.
   - ✓ $x \in X$, $E \in \mathcal{N}(x)$ implies $x \in \mathcal{N}(x)$ .
   - ✓ $x \in X, E \in \mathcal{N}(x)$ implies that $\exists F \in \mathcal{N}(x)$ such that $\forall y \in F$, $E \in \mathcal{N}(y)$.

   then there is a unique topology $\tau \in X$ , such that $\mathcal{N}_\tau(x) = \mathcal{N}(x)$ , for each $x \in X$ .

Let $\mathcal{P}(X \times X)$ be the collection of all subsets of $X \times X$. Any members of $\mathcal{P}(X \times X)$ is called a (binary) relation on $X$.

We denote by $\top$ the bijection:

$$\top : X \times X \to X \times X$$
$$\top(x, y) = (y, x).$$

For a $\mathrm{Q} \in \mathcal{P}(X \times X)$ we write $\top(\mathrm{Q}) = \{\top(x,y) : (x,y) \in \mathrm{Q}\}$ and call $\top(\mathrm{Q})$ the **conjugate** of $\mathrm{Q}$.

Usually in the literature the notation $\mathrm{Q}^{-1}$ is used instead of $\top(\mathrm{Q})$. The relation $\top(\mathrm{Q})$ is called also converse relation to $\mathrm{Q}$ (see Clifford-Preston (1961), p.14).

For $\mathrm{P}, \mathrm{Q} \subset X \times X$ we write

$$\mathrm{P} \circ \mathrm{Q} := \{(x, y) \in X \times X : \exists z \in X, (x, z) \in \mathrm{Q}, (z, y) \in \mathrm{P}\}$$

The relation $\mathrm{P} \circ \mathrm{Q}$ is called the **composition** of $P$ and $\mathrm{Q}$.
In $\mathcal{P}(X \times X)$ the composition $\circ$ can be viewed as a binary operation.

We write $\Delta = \Delta_X = \{(x, x) \text{ with } x \in X\}$, and call $\Delta$ the diagonal set.
A relation $\mathrm{Q}$ is called:

- ✓ reflexive iff $\Delta \subset \mathrm{Q}$.
- ✓ Symmetric iff $\top(\mathrm{Q}) = \mathrm{Q}$.
- ✓ anti-symmetric iff $\mathrm{Q} \cap \top(\mathrm{Q}) = \Delta$.
- ✓ Transitive iff $\mathrm{Q} \circ \mathrm{Q} \subset \mathrm{Q}$.

For a relation $\mathrm{Q} \in \mathcal{P}(X \times X)$, and an element $x \in X$, we define the (**vertical cross-)sections** of $\mathrm{Q}$ at $x$ as follows:

$$\mathrm{Q}[x] := \{y \in X : (x, y) \in \mathrm{Q}\}$$

and the **(vertical cross-)sections** at $A \subset X$, by the equality:

$$\text{Q}[A] := \bigcup_{x\in A} \text{Q}[x].$$

## 3. Uniform type structure

Let $X$ be a non-empty set and $\mathcal{Q}$ be a filter on $X \times X$ consisting of reflexive relations $(\Delta \subset \text{Q}, \quad \forall \text{Q} \in \mathcal{Q})$, we say that $\mathcal{Q}$ is a:

**Local Quasi-uniformity** if

$$\forall x \in X, \ \forall \text{Q} \in \mathcal{Q} \quad \exists \, \text{P} \in \mathcal{Q} \text{ such that } \text{P} \circ \text{P} \, [x] \subset \text{Q}[x].$$

**Local Uniformity** if

$$\forall x \in X, \forall \text{Q} \in \mathcal{Q}, \quad \top(\text{Q}) \in \mathcal{Q} \text{ and }, \ \exists \, \text{P} \in \mathcal{Q} \text{ such that } \text{P} \circ \text{P}[x] \subset \text{Q}[x].$$

**Quasi-Uniformity** if

$$\forall \text{Q} \in \mathcal{Q} \quad \exists \, \text{P} \in \mathcal{Q} \text{ such that } \text{P} \circ \text{P} \subset \text{Q}.$$

**Uniformity** if

$$\forall \text{Q} \in \mathcal{Q} \quad \exists \, \text{P} \in \mathcal{Q} \text{ such that } \top(\text{P}) \circ \text{P} \subset \text{Q}.$$

The pair $(X, \mathcal{Q})$ is called a local quasi-uniform space (resp. local uniform space, quasi-uniform space, uniform space) when $\mathcal{Q}$ is a local quasi-uniformity (resp. local uniformity, quasi-uniformity, uniformity) and the members of $\mathcal{Q}$ will be called entourages [2].

**Example 3.1**
Let $X$ be a non empty set. We say that a mapping $\rho : X \times X \to [0, +\infty[$ is a pseudo-quasi-metric if it satisfies:

- ✓ $\rho(x, x) = 0, \forall x \in X$.
- ✓ $\rho(x, y) \leq \rho(x, z) + \rho(z, y), \forall x, y, z \in X$.

The pair $(X, \rho)$ will be called pseudo-quasi-metric space.

Consider in $X \times X$ the family

$$\mathcal{Q}_\rho = \{\text{Q}_\rho(\varepsilon) \text{ with } \varepsilon > 0\} \text{ with } \text{Q}_\rho(\varepsilon) = \{(x, y) \in X \times X : \rho(x, y) < \varepsilon\}.$$

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[2]Some authors use the term "vicinity" instead of entourage (see Picado (1998)).

It's easy to see that:

- ✓ $Q_\rho$ is a filter base.
- ✓ $\rho(x,x)=0, \forall x \in X$ is equivalent to $\Delta \subset Q_\rho(\varepsilon), \forall \varepsilon > 0$.
- ✓ $\rho(x,y) \le \rho(x,z)+\rho(z,y), \forall x,y,z \in X$ implies that:

$$\forall \varepsilon > 0, \exists \delta > 0, \text{ such that } Q_\rho(\delta) \circ Q_\rho(\delta) \subset Q_\rho(\varepsilon).$$

For a pseudo-quasi-metric, $\rho$ , $\mathbb{F}(Q_\rho)$ is a quasi.-uniformity.

In following remark, we are giving equivalent characterizations of local uniformities and uniformities.

**Remark 3.2**
Let $X$ be a non-empty and, $Q$ be a filter on $X \times X$ , consisting of reflexive relations.

1. The following statements are equivalent.
   - ✓ $Q$ is a local uniformity.
   - ✓ $\forall Q \in Q \;\; \top(Q) \in Q$ and $\forall x, \forall Q \in Q \;\; \exists P \in Q : \top(P) \circ P[x] \subset Q[x]$.
2. The following statements are equivalent
   - ✓ $Q$ is a uniformity.
   - ✓ $\forall Q \in Q \;\; \exists S$ symmetric entourage such that $S \circ S \subset Q$.
   - ✓ $\forall Q \in Q \;\; \top(Q) \in Q$ and $\forall Q \in Q \; \exists \, P \in Q : \; P \circ P \subset Q$.

The next remark gives some properties of quasi-uniform space.
**Remark 3.3**
If $(X,Q)$ is a quasi-uniform space then:

1. $(X,Q)$ is a local quasi-uniform space.
2. The filter $Q^\top = \{\top(Q)$ such that $Q \in Q\}$ is quasi-uniformity. It is called the **conjugate quasi-uniformity** of $Q$ .
3. $Q$ is an uniformity if and only if $Q = Q^\top$.

**Remark 3.4**
If $(X,Q)$ is a local quasi-uniform space then:

1. The filter $Q^\top$ may not be a local quasi-uniformity, (see example 3.7).
2. $Q^\top$ may be a local quasi-uniformity, but $Q$ may not be a quasi-

uniformity (see example 3.8).

We say that $Q$ is a **bilocal quasi-uniformity** if $Q$ is a local quasi-uniformity which $Q^{\top}$ is a local **quasi-uniformity** too, and we that the pair $(X,Q)$ is a **bilocal quasi-uniform space**.

**Remark 3.5**
If $Q$ is a bilocal quasi-uniformities, then $Q$ is a local uniformity if and only if $Q = Q^{\top}$.

We say that a family $\mathcal{B}$ on $X \times X$ is a:

**Local quasi-uniform base** if it satisfies the following conditions:

$b_1^{'}$) $\mathcal{B}$ is a filter basis.

$b_2^{'}$) $\Delta \subset B, \quad \forall \mathrm{B} \in \mathcal{B}$.

$b_3^{'}$) $\forall x \in X$, $\forall \mathrm{B} \in \mathcal{B}$ $\exists \mathrm{C} \in \mathcal{B}$ such that $\mathrm{C} \circ \mathrm{C}[x] \subset \mathrm{B}[x]$.

**Local uniform base** if it satisfies the following conditions:

$b_1^{'}$) $\mathcal{B}$ is a filter basis.

$b_2^{'}$) $\Delta \subset B, \quad \forall \mathrm{B} \in \mathcal{B}$.

$b_3^{'}$) $\forall \mathrm{B} \in \mathcal{B}$ $\exists \mathrm{C} \in \mathcal{B}$ such that $\mathrm{C} \subset \top(\mathrm{B})$

$b_4^{'}$) $\forall x \in X$, $\forall \mathrm{B} \in \mathcal{B}$ $\exists \mathrm{C} \in \mathcal{B}$ such that $\mathrm{C} \circ \mathrm{C}[x] \subset \mathrm{B}[x]$.

**Quasi-uniform base** if it satisfies the following conditions:

$b_1$) $\mathcal{B}$ is a filter basis.

$b_2$) $\Delta \subset \mathrm{B} \quad \forall \mathrm{B} \in \mathcal{B}$.

$b_3$) $\forall B \in \mathcal{B} \quad \exists C \in \mathcal{B}$ such that $C \circ C \subset B$.

**Uniform base** if it satisfies the following conditions:

$b_1$) $\mathcal{B}$ is a filter basis.

$b_2$) $\Delta \subset B, \quad \forall B \in \mathcal{B}$.

$b_3$) $\forall B \in \mathcal{B} \quad \exists C \in \mathcal{B}$ such that $C \subset \top(B)$

$b_4$) $\forall B \in \mathcal{B} \quad \exists C \in \mathcal{B}$ such that $C \circ C \subset B$.

If $\mathcal{B}$ is a (local) quasi-uniform base, then $\mathbb{F}(\mathcal{B})$[3] is a unique (local) quasi-uniformity for which $\mathcal{B}$ is base.

A family $\mathcal{S} \subset \mathcal{P}(X \times X)$ is a **subbase** of a (local) quasi-uniformty, if the family $\mathcal{B}$ of finite intersections of members of $\mathcal{S}$ is a (local) quasi-uniform base.

**Example 3.6**

1. For a set $X$ and its subset $G$ we write:

$$S_G = (G \times G) \cup ((X - G) \times X) \subset X \times X .$$

   It is easy to observe that $S_G$ is a reflexive relation with the property: $S_G \circ S_G = S_G$.

Let $(X, \tau)$ be a topological space. $\mathcal{S} = \{S_G : G \in \tau\}$ is a quasi-uniform subbase (this is easy to see). The quasi-uniformity generated by this family is called by **Pervin quasi-uniformity** associated with the topology $\tau$, and it is denoted by $Q^{Per}(\tau)$ (see Murdeshwar- Naimpally(1966)).

[3] $\mathbb{F}(\mathcal{B})$ is the filter generated by the filter base $\mathcal{B}$.

**Example 3.7**

Write $X = \{0\} \cup \{\frac{1}{n}\}$, with $n \in \mathbb{N}$ and

$$Q_n = \Delta \cup \{(\frac{1}{i}, 0) : i \geq n\} \cup \{(\frac{1}{i+1}, \frac{1}{i}) : i \geq n\} \cup \{(1, \frac{1}{i}) : i \geq n\}$$

and $\mathcal{Q}_o = \{Q_n : n \in \mathbb{N}\}$.

It's easy to see that:

- ✓ $Q_{n+1} \subset Q_n$, for each $n \in \mathbb{N}$, therefore $\mathcal{Q}_o$ is a filter base on $X \times X$.
- ✓ $\Delta \subset Q_n$ for every $n \in \mathbb{N}$.

- Let us see that $\mathcal{Q}_o$ isn't a quasi-uniformity base.

Fix $Q_1 \in \mathcal{Q}_o$ and the pair $(\frac{1}{n'+2}, \frac{1}{n'})$.

We have

$$\left(\frac{1}{n'+2}, \frac{1}{n'+1}\right) \in Q_{n'} \text{ and } \left(\frac{1}{n'+1}, \frac{1}{n'}\right) \in Q_{n'}, \forall n' \in \mathbb{N}.$$

Then for every $n' \in \mathbb{N}$, $(\frac{1}{n'+2}, \frac{1}{n'}) \in Q_{n'} \circ Q_{n'}$, however $(\frac{1}{n'+2}, \frac{1}{n'}) \notin Q_1$.

- Let us see that $\mathcal{Q}_o$ is a local quasi-uniformity base.

Fix $n \in \mathbb{N}$, we have

$$Q_n[0] = Q_n \circ Q_n[0] = \{0\} \quad \forall n \in \mathbb{N};$$

$$Q_1[1] = Q_1[1] \circ Q_1[1] = \left\{0, 1, \tfrac{1}{2}, \ldots\right\};$$

$$Q_n[1] = Q_n \circ Q_n[1] = \{1, \tfrac{1}{n}, \tfrac{1}{n+1}, \ldots\}, \quad \forall n > 1.$$

Fix $n \in \mathbb{N}$ and $k > 1$ then:

$$Q_{k+1} \circ Q_{k+1}[\frac{1}{k}] = \{\frac{1}{k}\} \text{ hence } Q_{k+1} \circ Q_{k+1}[\frac{1}{k}] \subset Q_n[\frac{1}{k}], \forall n \in \mathbb{N}.$$

- Now let us see that $\mathcal{Q}^{\top}$ isn't a local quasi-uniformity base.

Observe that: $\top(Q_n) = \Delta \cup \{(0, \frac{1}{i})\}_{i \geq n} \cup \{(\frac{1}{i}, \frac{1}{i+1})\}_{i \geq n} \cup \{(\frac{1}{i}, 1)\}_{i \geq n}$.

Then $\top(Q_2)[0] = \{0, \frac{1}{2}, \frac{1}{3}, \ldots\}$ but for every $m \in \mathbb{N}$, we have:

$$\left(0, \frac{1}{m}\right) \in Q_m \text{ and } \left(\frac{1}{m}, 1\right) \in Q_m \text{, thus}$$

$$1 \in \top(Q_m) \circ \top(Q_m)[0], \forall m \in \mathbb{N}, \text{ but } 1 \notin \top(Q_2)[0].$$

Next we present the example mentioned in remark 3.4 (2).

**Example 3.8**

Write $X = \{0\} \cup \{\frac{1}{n}\}$, with $n \in \mathbb{N}$, and

$$Q_n = \Delta \cup \{(0, \tfrac{1}{i}) : i \geq n\} \cup \{(\tfrac{1}{i+1}, \tfrac{1}{i}) : i \geq n\}$$

and $\mathcal{Q}_o = \{Q_n : n \in \mathbb{N}\}$

Let us see that $\mathcal{Q}_o$ isn't a quasi-uniformity base.

It's easy to see that:

- ✓ $Q_{n+1} \subset Q_n$ for every $n \in \mathbb{N}$, therefore $\mathcal{Q}$ is a filter base on $X \times X$.
- ✓ $\Delta \subset \mathcal{Q}_n$ for every $n \in \mathbb{N}$.

- Let us see that $\mathcal{Q}$ isn't a quasi-uniformity base.

Fix $Q_1 \in \mathcal{Q}_o$ and $n' \in \mathbb{N}$. We have:

$$\left(\frac{1}{n'+2}, \frac{1}{n'+1}\right) \in Q_{n'} \text{ and } \left(\frac{1}{n'+1}, \frac{1}{n'}\right) \in Q_{n'}.$$

Then for every $n' \in \mathbb{N}$, we have:

$$\left(\frac{1}{n'+2}, \frac{1}{n'}\right) \in Q_{n'} \circ Q_{n'}, \text{ but } \left(\frac{1}{n'+2}, \frac{1}{n'}\right) \notin Q_1.$$

- Now let us see that $\mathcal{Q}$ is a local quasi-uniformity base.

For each $n \in \mathbb{N}$, we have:

$$Q_n[0] = Q_n \circ Q_n[0] = \{0, \frac{1}{n}, \frac{1}{n+1} \ldots\}$$

Now let $k \geq 1$, and $n \geq 1$. We have:

$$Q_k \circ Q_k[\frac{1}{k}] = \{\frac{1}{k}\} \text{ hence } Q_k \circ Q_k[\frac{1}{k}] \subset Q_n[\frac{1}{k}].$$

- Let us see that $\mathcal{Q}^{\top}$ is also a local quasi-uniformity base.

Notice that $\top(Q_n) = \{\Delta\} \cup \{(\frac{1}{i}, 0) : i \geq n\} \cup \{(\frac{1}{i}, \frac{1}{i+1}) : i \geq n\}$.

Since $Q$ is a filter base of reflexive relations, then $Q^{\top}$ is a filter base of reflexive relations, too.

For any $n \in \mathbb{N}$, we know that:

$$\top(\mathrm{Q}_n)[0] = \top(\mathrm{Q}_n) \circ \top(\mathrm{Q}_n)[0] = \{0\}.$$

Now fix $n, k \in \mathbb{N}$, we have:

$$\top(\mathrm{Q}_{k+1}) \circ \top(\mathrm{Q}_{k+1})[\frac{1}{k}] = \{\frac{1}{k}\} \text{ hence } \top(\mathrm{Q}_{k+1}) \circ \top(\mathrm{Q}_{k+1})[\frac{1}{k}] \subset \top(\mathrm{Q}_n)[\frac{1}{k}].$$

In this example we introduce several an example of a uniform base and a quasi-uniform base.

**Example 3.9**

a) Let $X = \mathbb{R}$. For $\varepsilon > 0$ denote $\mathrm{U}_\varepsilon := \{(x, y) \in \mathbb{R}^2 : y - x \in ]-\varepsilon, \varepsilon[\}$. The uniformity generated by the base $\{\mathrm{U}_\varepsilon : \varepsilon > 0\}$ is called **the usual uniformity** of the real line $\mathbb{R}$.

b) Let $X = \mathbb{R}$. For $\varepsilon > 0$ denote $\mathrm{Q}_\varepsilon := \{(x, y) \in \mathbb{R}^2 : y - x \in [0, \varepsilon[\}$, then $\{\mathrm{Q}_\varepsilon : \varepsilon > 0\}$ is a quasi-uniform base. The quasi-uniformity **S** in $\mathbb{R}$ generated by this base is the quasi-uniformity induced by the **Sorgenfrey** topology.

We say that a local quasi-uniformity $Q$ is:

**weakly locally symmetric** at $x \in X$ if for every $\mathrm{Q} \in Q$ there is a symmetric entourage $S \in Q$ such that $S[x] \subset \mathrm{Q}[x]$.

**weakly locally symmetric** or **point-symmetric** if $Q$ is weakly locally symmetric at $x$, for every $x \in X$.

**small-set symmetric** at $x \in X$, if $(X, Q)$ is a bilocal quasi-uniformity $Q^{\top}$ is weakly locally symmetric at $x \in X$.

**locally symmetric** at $x \in X$ if for every $\mathrm{Q} \in Q$ there is a symmetric $S \in Q$ such that $S \circ S[x] \subset \mathrm{Q}[x]$.

**locally symmetric** if $Q$ is locally symmetric at $x$, for every $x \in X$.

The following example of Fletcher-Lindgren (1982) gives us a quasi-uniformity locally symmetric $Q$ which isn't uniformity.

**Example 3.10**

Let $X = \{0\} \cup \{\frac{1}{n}\}$, with $n \in \mathbb{N}$ and

$$Q = \{Q_n : n \in \mathbb{N}\} \text{ with } Q_n = \Delta \cup \{(\frac{1}{i}, 0)\}_{i \geq n}.$$

**Remark 3.11**

1. We can find a quasi-uniform version of the last definition in Künzi (2001), but in that case we must assume that $Q^{\top}$ is a local quasi-uniformity as well. In quasi-uniform case this condition always holds.

2. We can observe certain similarity between the notions of local uniformity and locally symmetric quasi-uniformity. However these concepts are different, because a local uniformity may not be divisible, and a quasi-uniformity locally symmetric may not contain the converse of any entourage.

**Proposition 3.12**

Let $(X, Q)$ be a local quasi-uniform space. $Q$ is weakly locally symmetric if and only if for any $x \in X$ and $Q \in Q$ there is an entourage $P \in Q$ such that $\top(P)[x] \subset Q[x]$.

**Proof:**

Suppose that for each $Q \in Q$ there is an entourage $P_1$ such that $\top(P_1)[x] \subset Q[x]$. Put $P = Q \cap P_1$ and we will consider the symmetric entourage $S = P \cup \top(P)$. Then $S[x] = \top(P)[x] \cup P[x] \subset Q[x]$. The reciprocal is immediate. ♦

**Proposition 3.13**

Let $(X, Q)$ be a local quasi-uniform space. $Q$ is a locally symmetric if and only if for any $x \in X$ and for any $Q \in Q$ there is an entourage $P \in Q$ such that $\top(P) \circ P[x] \subset Q[x]$.

**Proof:**

The first implication is obvious. If $\mathrm{Q} \in \mathcal{Q}$ and $x \in X$, there are entourages $\mathrm{P}_1$ and $\mathrm{P}_2$ such that $\top(\mathrm{P}_1) \circ \mathrm{P}_1[x] \subset \mathrm{Q}[x]$ and $\mathrm{P}_2 \circ \mathrm{P}_2[x] \subset \mathrm{Q}[x]$. If $\mathrm{P} = \mathrm{P}_1 \cap \mathrm{P}_2$, there is an entourage $R_1$ such that $\top(R_1) \circ R_1[x] \subset P[x]$.

If $R = R_1 \cap P \subset P$, and $S = \top(R) \cup R$, let us see that $S \circ S[x] \subset \mathrm{Q}[x]$.

We have:

i $\top(R) \circ \top(R)[x] \subset \top(R) \circ \top(R) \circ R[x] \subset \top(R) \circ (P)[x] \subset \top(P) \circ P[x] \subset \mathrm{Q}[x]$;

ii $R \circ R[x] \subset P \circ P[x] \subset \mathrm{Q}[x]$;

iii $\top(R) \circ R[x] \subset P[x] \subset P \circ P[x] \subset \mathrm{Q}[x]$;

iv. $R \circ \top(R)[x] \subset R \circ \top(R) \circ R[x] \subset R \circ (P)[x] \subset P \circ P[x] \subset \mathrm{Q}[x]$;

Consequently:

$$\big(\top(R) \cup R\big) \circ \big(\top(R) \cup R\big)[x] =$$
$$= \big(\top(R) \circ \top(R)\big)[x] \cup \big(\top(R) \circ R\big)[x] \cup \big(R \circ \top(R)\big)[x] \cup \big(R \circ R\big)[x] \subset \mathcal{Q}[x]. \blacklozenge$$

Let $X$ be a set and $\mathbf{LQ}(X)$ be the set of all local quasi-uniformities in $X$. Then:

— $\mathbf{LQ}(X)$ with respect to the set-theoretic inclusion $\subset$ is a partially ordered set.

— In $(\mathbf{LQ}(X), \subset)$ **the indiscrete uniformity** $\{X \times X\}$ is the smallest element and **the discrete uniformity** $\{R \in P(X \times X) | R \supset \Delta\}$ is the biggest element.

— $(\mathbf{LQ}(X), \subset)$ is a complete lattice.

Let $(\mathcal{Q}_i)_{i \in I}$ be a non-empty family of local quasi-uniformities (resp. family of quasi-uniformity) on $X$. We denote:

1. $\inf_{i \in I}(\mathcal{Q}_i) = \wedge_{i \in I} \mathcal{Q}_i$ is the finest local quasi-uniformity (resp. family of quasi-uniformity) contained $\mathcal{Q}_i$, $\forall i \in I$.
2. $\sup_{i \in I}(\mathcal{Q}_i) = \vee_{i \in I} \mathcal{Q}_i$ is the coarsest local quasi-uniformity (resp. family of quasi-uniformity) containing $\mathcal{Q}_i$, $\forall i \in I$.

**Remark 3.14**

Let $(\mathcal{Q}_i)_{i \in I}$ be a non-empty family of local quasi-uniformity (resp. family of quasi-uniformity) on $X$. Then

1. $\wedge_{i \in I} \mathcal{Q}_i \subset \mathcal{Q}_i$ and if there is a local quasi-uniformity (resp. a quasi-uniformity) $\mathcal{Q}$ such that for any $i$ we have $\mathcal{Q} \subset \mathcal{Q}_i$, then $\mathcal{Q} \subset \wedge_{i \in I} \mathcal{Q}_i$.

2. $\mathcal{Q}_i \subset \vee_{i\in I}\mathcal{Q}_i$ and if there is a local quasi-uniformities (resp. family of quasi-uniformities) $\mathcal{Q}$ such that $\mathcal{Q}_i \subset \mathcal{Q}, \forall i \in I$ then $\vee_{i\in I}\mathcal{Q}_i \subset \mathcal{Q}$.
3. $\wedge_{i\in I}\mathcal{Q}_i := \vee_{j\in J}\mathcal{P}_j$, where $(\mathcal{P}_j)_{j\in J}$ denotes the family of all local quasi-uniformities (resp. quasi-uniformities) contained in $\cap_{i\in I}\mathcal{Q}_i$ (the family $(\mathcal{P}_j)_{j\in J}$ is not empty, because at least the indiscrete uniformity is presents in $(\mathcal{P}_j)_{j\in J}$).

The following results given us a characterization about the local quasi-uniformities $\inf_{i\in I}(\mathcal{Q}_i)$ and $\sup_{i\in I}(\mathcal{Q}_i)$.

**Lemma 3.15**

Let $X$ be a non-empty set and $(\mathcal{Q}_i)_{i\in I}$ be a family of local quasi-uniformity (resp. quasi-uniformity ). Then:

1. $\{\cap_{i\in I_0} \mathrm{Q}_i : \mathrm{Q}_i \in \mathcal{Q}_i, I_0 \text{ is finite}\}$ is a base for $\vee_{i\in I}\mathcal{Q}_i$.
2. (a) $\wedge_{i\in I}\mathcal{Q}_i \subset \cap_{i\in I}\mathcal{Q}_i$.

   (b) $\cap_{i\in I}\mathcal{Q}_i$ is local quasi-uniformity (resp. quasi-uniformity) then $\wedge_{i\in I}\mathcal{Q}_i = \cap_{i\in I}\mathcal{Q}_i$ and $\mathcal{B} = \{\cup_{i\in I}\mathrm{Q}_i : \mathrm{Q}_i \in \mathcal{Q}_i\}$ is a base of $\wedge_{i\in I}\mathcal{Q}_i$.

**Proof:**

1. It is easy to prove.

2 (a) It's obvious.

(b) It is enough to show that $\cap_{i\in I}\mathcal{Q}_i \subset \wedge_{i\in I}\mathcal{Q}_i$.

For each $i \in I$, we know that $\cap_{i\in I}\mathcal{Q}_i \subset \mathcal{Q}_i$.

Therefore by definition of least lower bound we have $\cap_{i\in I}\mathcal{Q}_i \subset \wedge_{i\in I}\mathcal{Q}_i$.

It is easy to prove that $B = \{\cup_{i\in I}\mathrm{Q}_i : \mathrm{Q}_i \in \mathcal{Q}_i\}$ is base of $\cap_{i\in I}\mathcal{Q}_i$. ♦

If $\wedge_{i\in I}\mathcal{Q}_i = \cap_{i\in I}\mathcal{Q}_i$, then we have a description of a base for $\wedge_{i\in I}\mathcal{Q}_i$. But in general that doesn't happen as we can verify in the example 4.8.

**Lemma 3.16**

Let $\{\mathcal{Q}_1,...,\mathcal{Q}_n\}$ be a finite family of quasi-uniformity. If the family

$$\mathcal{B} := \{Q_1 \circ ... \circ Q_n : Q_i \in \mathcal{Q}_i, \forall i = 2, ...., n\}$$

is a base of one quasi-uniformity $\mathcal{Q}$, then $\mathcal{Q} = \wedge_{i=1}^{n} \mathcal{Q}_i$.

**Proof:**

Evidently, for each $i$ for $i = 1, ..., n$ we have $\mathcal{Q} \subset \mathcal{Q}_i$, and $\mathcal{Q} \subset \bigcap_{i=1}^{n} \mathcal{Q}_i$. The definition of the greatest lower bound gives $\mathcal{Q} \subset \wedge_{i=1}^{n} \mathcal{Q}_i$. Let us show that $\wedge_{i=1}^{n} \mathcal{Q}_i \subset \mathcal{Q}$. Taking $U \in \wedge_{i=1}^{n} \mathcal{Q}_i$, there is $V \in \wedge_{i\in I} \mathcal{Q}_i$ such that $V \circ ... \circ V \subset U$. Since for each $i$, we have $V \in \mathcal{Q}_i$, therefore the set $V \circ ... \circ V$ belongs to $\mathcal{B}$, which is a base of $\mathcal{Q}$. Consequently $U \in \mathcal{Q}$. ♦

**Lemma 3.17**

Let $\{\mathcal{Q}_i\}_{i\in I}$ be a family of bilocal quasi-uniformities. Then $\left(\vee_{i\in I} \mathcal{Q}_i\right)^{\top} = \vee_{i\in I} \mathcal{Q}_i^{\top}$ and $\left(\wedge_{i\in I} \mathcal{Q}_i\right)^{\top} = \wedge_{i\in I} \mathcal{Q}_i^{\top}$.

**Proof:**

We will proof that $\left(\wedge_{i\in I} \mathcal{Q}_i\right)^{\top} = \wedge_{i\in I} \mathcal{Q}_i^{\top}$. We observe that for every $i$, we have $\wedge_{i\in I} \mathcal{Q}_i \subset \mathcal{Q}_i$ then $\left(\wedge_{i\in I} \mathcal{Q}_i\right)^{\top} \subset \mathcal{Q}_i^{\top}$. If for every $i$, there is a local-quasi-uniformity $\mathcal{V}$ such that $\mathcal{V} \subset \mathcal{Q}_i^{\top}$, then $\mathcal{V}^{\top} \subset \mathcal{Q}_i$ but by definition of nfimum $\mathcal{V}^{\top} \subset \wedge_{i\in I} \mathcal{Q}_i$, therefore $\mathcal{V} \subset \left(\wedge_{i\in I} \mathcal{Q}_i\right)^{\top}$. The proof in the case of the supremum is similar. ♦

**Lemma 3.18**

Let $X$ be a non-void set and $\mathcal{P}$, $\mathcal{Q}$ be local uniformities (resp. uniformities) on $X$, then $\mathcal{P} \vee \mathcal{Q}$ and $\mathcal{P} \wedge \mathcal{Q}$ are local uniformities (resp. uniformities) too.

**Proof:**

By lemma 3.15 it's easy to check that $\mathcal{P} \vee \mathcal{Q}$ is local uniformity (resp. uniformity). Put $\mathcal{V} = \mathcal{P} \wedge \mathcal{Q}$. Since $\mathcal{P}$ and $\mathcal{Q}$ are a local uniformities (resp. uniformities) by 3.5 (resp. 3.3(4)) $\mathcal{P} = \mathcal{P}^{\top}$ and $\mathcal{Q} = \mathcal{Q}^{\top}$ then $\mathcal{V} = \mathcal{P}^{\top} \wedge \mathcal{Q}^{\top}$, but by 3.17 we know that $\mathcal{P}^{\top} \wedge \mathcal{Q}^{\top} = (\mathcal{P} \wedge \mathcal{Q})^{\top} = \mathcal{V}$. Then $\mathcal{V} = \mathcal{V}^{\top}$ hence $\mathcal{V}$ is local uniformity (resp. uniformity). ♦

**Lemma 3.19**
Let $(X,\mathcal{Q})$ be a bilocal quasi-uniform space.

1. Put $\mathcal{Q}^{\vee} = \mathcal{Q} \vee \mathcal{Q}^{\top}$.
   (a) The family $\{\top(\mathrm{Q}) \cap \mathrm{Q} : \mathrm{Q} \in \mathcal{Q}\}$ is a local quasi-uniformity base for $\mathcal{Q}^{\vee}$.
   (b) $\mathcal{Q}^{\vee}$ is the coarsest local uniformity containing $\mathcal{Q}$.
2. Put $\mathcal{Q}_{\wedge} = \mathcal{Q} \wedge \mathcal{Q}^{\top}$.
   (a) If $\mathcal{Q} \cap \mathcal{Q}^{\top}$ is a local uniformity then $\mathcal{Q}_{\wedge} = \mathcal{Q} \cap \mathcal{Q}^{\top}$ and the family
   $\mathcal{B} = \{\top(\mathrm{Q}) \cup \mathrm{Q} : \ \mathrm{Q} \in \mathcal{Q}\}$ is base of $\mathcal{Q}_{\wedge}$.
   (b) $\mathcal{Q}_{\wedge}$ is the finest local uniformity contained in $\mathcal{Q}$.

**Proof:**

1. (a) It is a particular case of 3.15(a).
   (b) By $a)$ we have the family $\{\mathrm{Q} \cap \top(\mathrm{Q}) : \mathrm{Q} \in \mathcal{Q}\}$ is a base for $\mathcal{Q} \vee \mathcal{Q}^{\top}$. Therefore the all members of this family are symmetric, and we get that $\mathcal{Q} \vee \mathcal{Q}^{\top}$ is uniformity. The rest is clear.
2. (a) It is a particular case of 3.15 (b).
   (b) Put $\mathcal{V} := \mathcal{Q} \wedge \mathcal{Q}^{\top}$. Clearly, $\mathcal{V} \subset \mathcal{Q}$ and $\mathcal{V}^{\top} \subset \mathcal{Q}$. Then $\mathcal{W} := \mathcal{V} \vee \mathcal{V}^{\top} \subset \mathcal{Q}$, hence $\mathcal{V} \subset \mathcal{W}$. By the last point, $\mathcal{W}$ is a local uniformity, hence $\mathcal{W} = \mathcal{W}^{\top} \subset \mathcal{Q}^{\top}$. By the definition of $\mathcal{V}$, we get $\mathcal{W} \subset \mathcal{V}$, i.e. $\mathcal{V} = \mathcal{W}$ is a local uniformity. The rest is clear. ♦

The following corollary is a particular case of 3.19 and 3.16.

**Corollary 3.20**
Let $(X,\mathcal{Q})$ be a quasi-uniform space.

1. Put $\mathcal{Q}^{\vee} = \mathcal{Q} \vee \mathcal{Q}^{\top}$.
   (a) The family $\{\mathrm{Q} \cap \top(\mathrm{Q}) : \mathrm{Q} \in \mathcal{Q}\}$ it is a quasi-uniformity base for $\mathcal{Q}^{\vee}$.
   (b) Then $\mathcal{Q}^{\vee}$ is a uniformity and it is the coarsest uniformity containing $\mathcal{Q}$.

2. Put $\mathcal{Q}_{\wedge} = \mathcal{Q} \wedge \mathcal{Q}^{\top}$.
   (a) If the family $\mathcal{B} := \{\top(\mathrm{Q}) \circ \mathrm{Q} : \mathrm{Q} \in \mathcal{Q}\}$ or $\mathcal{B} := \{\mathrm{Q} \circ \top(\mathrm{Q}) : \mathrm{Q} \in \mathcal{Q}\}$ is a uniformity base then, it is a base for the uniformity $\mathcal{Q}_{\wedge}$.
   (b) If $\mathcal{Q} \cap \mathcal{Q}^{\top}$ is a quasi-uniformity then $\mathcal{Q}_{\wedge} = \mathcal{Q} \cap \mathcal{Q}^{\top}$ and the family $\mathcal{B} = \{\mathrm{Q} \cup \top(\mathrm{Q}) : \ \mathcal{Q} \in \mathcal{Q}\}$ is base of $\mathcal{Q}_{\wedge}$
   (c) Then $\mathcal{Q}_{\wedge}$ is a uniformity and it is the finest uniformity contained in $\mathcal{Q}$.

**Remark 3.21**
The families $\{\top(\mathrm{Q}) \circ \mathrm{Q}\colon \mathrm{Q} \in \mathcal{Q}\}$, $\{\mathrm{Q} \circ \top(\mathrm{Q})\colon \mathrm{Q} \in \mathcal{Q}\}$ are always filter bases, and each element of this family is symmetric and containing the diagonal, but can't be a quasi-uniformity basis.

## 4. Topologies defined in uniform type structures

We begin this chapter with the next result:

**Proposition 4.1.**
Let $X$ be a non empty set and $\mathcal{Q}$ be a filter on $X \times X$, then the family

$$\tau_{\mathcal{Q}} = \{A \subset X : \forall a \in A, \exists \mathrm{Q} \in \mathcal{Q} \text{ such that } \mathrm{Q}[a] \subset A\}$$

is a topology on $X$, which will be called the topology induced by $\mathcal{Q}$.

**Proof:**
It's easy to see that $\emptyset,\ X \in \tau_{\mathcal{Q}}$. Let $\{A_i\}_{i\in I}$ a family of elements of $\tau_{\mathcal{Q}}$, and let us see that $\bigcup_{i\in I} A_i \in \tau_{\mathcal{Q}}$. Fix $x \in \bigcup_{i\in I} A_i$, then there is a $i_0$ such that $x \in A_{i_0}$. By definition of $\tau_{\mathcal{Q}}$, there is a $\mathrm{Q} \in \mathcal{Q}$ such that $\mathrm{Q}[x] \subset A_{i_0}$, thus $\mathrm{Q}[x] \subset \bigcup_{i\in I} A_i$.

Let now $A_1, A_2 \in \tau_{\mathcal{Q}}$ we need to show that $A_1 \cap A_2 \in \tau_{\mathcal{Q}}$; take $x \in A_1 \cap A_2$, then there are $\mathrm{Q}_1,\ \mathrm{Q}_2 \in \mathcal{Q}$ such that $\mathrm{Q}_1[x] \subset A_1$ and $\mathrm{Q}_2[x] \subset A_2$, since $\mathrm{Q}_1 \cap \mathrm{Q}_2 \in \mathcal{Q}$ we get $(\mathrm{Q}_1 \cap \mathrm{Q}_2)[x] \subset \mathrm{Q}_1[x] \cap \mathrm{Q}_2[x] \subset A_1 \cap A_2$. ♦

In general if $\mathcal{Q}$ is a filter on $X \times X$, it may happen that for a given $\mathrm{Q} \in \mathcal{Q}$ and $x \in X$ the set $\mathrm{Q}[x]$ is not a $\tau_{\mathcal{Q}}$-neighbourhood of $x$.

**Example 4.2**

Let $X = \{1,2,3\}$ and $\mathrm{Q} = \{(1,1),(2,2),(3,3),(2,3),(3,1)\}$ and we consider the filter $\mathcal{Q} := \mathbb{F}(\{\mathrm{Q}\})$.

It is easy to see that $\tau_{\mathcal{Q}} = \{\{\emptyset\},\{1\},\{1,3\},\{1,2,3\}\}$, but $\mathcal{Q}[2] \notin \mathcal{N}_{\tau_{\mathcal{Q}}}(2)$.

From the next proposition we will derive that the phenomenon from the above example cannot happen when $\mathcal{Q}$ is a local quasi-uniformity (see corollary 4.4)

**Proposition 4.3**
Let $\mathcal{Q}$ be a local quasi-uniformity, $A$ be a subset of $X$ and $\widetilde{A} = \{x \in A, \exists\, \mathrm{Q} \in \mathcal{Q} : \mathrm{Q}[x] \subset A\}$. Then $\widetilde{A}$ is the interior of $A$ for a topology $\tau_{\mathcal{Q}}$.

**Proof:**

First we are going to prove that $\widetilde{A} \in \tau_{\mathcal{Q}}$. Take $x \in \widetilde{A}$. There is a $\mathrm{Q} \in \mathcal{Q}$ such that $\mathrm{Q}[x] \subset A$. Since $\mathcal{Q}$ is a local quasi-uniformity, there is a $\mathrm{P} \in \mathcal{Q}$ such that $\mathrm{P} \circ \mathrm{P}[x] \subset \mathrm{Q}[x]$.

Let us see that $\mathrm{P}[x] \subset \widetilde{A}$. It is enough to prove that $\mathrm{P}[y] \subset A$, for every $y \in \mathrm{P}[x]$.
Let $z \in \mathrm{P}[y]$ then $(y,z) \in \mathrm{P}$, since $y \in \mathrm{P}[x]$ therefore $(x,y) \in \mathrm{P}$ then $(x,z) \in \mathrm{P} \circ \mathrm{P}$, but

$$z \in \mathrm{P} \circ \mathrm{P} \subset \mathrm{Q}[x], \text{ hence } \mathrm{P}[y] \subset \mathrm{Q}[x] \subset A.$$

Since $\widetilde{A}$ is an open set such that $\widetilde{A} \subset A$, we know that $\widetilde{A} \subset int(A)$.

Now, we want to prove that $int(A) \subset \widetilde{A}$. Let $a \in int(A)$ then there is a $\mathrm{Q}$ such that $\mathrm{Q}[a] \subset int(A) \subset A$ then $int(A) \subset \widetilde{A}$, hence $\widetilde{A} = int(A)$. ♦

**Corollary 4.4**
Let $(X, \mathcal{Q})$ be a local quasi-uniform space. Then

$$\mathcal{N}_{\tau_{\mathcal{Q}}}(x) = \{\mathrm{Q}[x],\ \mathrm{Q} \in \mathcal{Q}\}, \quad \forall x \in X.$$

**Proof:**

Fix $x \in X$, $\mathrm{Q} \in \mathcal{Q}$ and let us see that $\mathrm{Q}[x] \in \mathcal{N}_{\tau_Q}(x)$. In fact, by the proposition 4.3 $\widetilde{\mathrm{Q}}[x] \in \tau_{\mathrm{Q}}$ and $\widetilde{\mathrm{Q}}[x] \subseteq \mathrm{Q}[x]$. Evidently $x \in \widetilde{\mathrm{Q}}[x]$. Consequently $\mathrm{Q}[x] \in \mathcal{N}_{\tau_Q}(x)$. Therefore we proved that $\{\mathrm{Q}[x],\ \mathrm{Q} \in \mathcal{Q}\} \subset \mathcal{N}_{\tau_Q}(x),\ \ \forall x \in X$.

Let us show now that $\{\mathrm{Q}[x],\ \mathrm{Q} \in \mathcal{Q}\}$ is a base of $\mathcal{N}_{\tau_Q}(x)\ \ \forall x \in X$.

Take $x \in X$ and $\mathrm{E} \in \mathcal{N}_{\tau_Q}(x)$, we need to find $\mathrm{Q} \in \mathcal{Q}$ such that $\mathrm{Q}[x] \subset E$. Since $\mathrm{E} \in \mathcal{N}_{\tau_Q}(x)$ there is a $\tau_Q$-open $G$ such that $x \in G \subset E$ then by the definition of $\tau_Q$ there is a $\mathrm{Q} \in \mathcal{Q}$ such that $\mathrm{Q}[x] \subset G$. Consequently $\mathrm{Q}[x] \subset G \subset E$.

It remains to show that $\mathcal{N}_{\tau_Q}(x) \subset \{\mathrm{Q}[x],\ \mathrm{Q} \in \mathcal{Q}\},\ \ \forall x \in X$. Take $x \in X$ and $\mathrm{E} \in \mathcal{N}_{\tau_Q}(x)$, we need to find $\mathrm{P} \in \mathcal{Q}$ such that $E = \mathrm{P}[x]$. Since $\{\mathrm{Q}[x],\ \mathrm{Q} \in \mathcal{Q}\}$ is a base of $\mathcal{N}_{\tau_Q}(x)$ there is $\mathrm{Q} \in \mathcal{Q}$ such that $\mathrm{Q}[x] \subset E$

Write $P := \mathrm{Q} \cup (E \times E)$, since $\mathcal{Q}$ is a filter, $\mathrm{P} \in \mathcal{Q}$ it is clear that $P[x] \subset E$. ♦

We can see in Murdeshwar-Naimpally (1966), (pg. 11) that the family $\mathcal{N}_{\tau_Q}(x) = \{\mathrm{Q}[x],\ \mathrm{Q} \in \mathcal{Q}\}$ satisfies the Hausdorff conditions, then we can say that there is only one topology $\tau$ such that for each $x \in X$, the family of all neighbours at $x$ is $\mathcal{N}_{\tau_Q}(x) = \{\mathrm{Q}[x],\ \mathrm{Q} \in \mathcal{Q}\}$. It's easy to prove the same result for the local quasi- uniform spaces.

**Examples 4.5.**
The quasi-uniformity bases of the example 3.9 induce different topologies, like this:

- ✓ The usual uniformity $\mathcal{E}$ on ℝ induces the usual (or Euclidean) topology **e** on ℝ; clearly, for a given $x \in X$ the family $\{U_{\mathcal{E}}[x]{:}\varepsilon{>}0\} = \{]x-\varepsilon, x+\varepsilon[, \varepsilon > 0\}$ is a base of $\mathcal{N}_{\mathcal{E}}(x)$.

✓ The quasi-uniformity base **S** on ℝ (example b)) induces the Sorgenfrey topology $\sigma$ ; for a given $x \in X$ the family $\{Q_\varepsilon[x]: \varepsilon > 0\} = \{[x, x+\varepsilon[, \varepsilon > 0\}$ is a base of $\mathcal{N}_\sigma(x)$.

**Lemma 4.6**

Let $(\mathcal{Q}_i)_{i\in I}$ be a non-empty family of local quasi-uniformities on X.

a) $\tau_{\vee_{i\in I}\mathcal{Q}_i} = \vee_{i\in I}\tau_{\mathcal{Q}_i}$.

b) $\tau_{\cap_{i\in I}\mathcal{Q}_i} \subset \cap_{i\in I}\tau_{\mathcal{Q}_i}$.

c) $\tau_{\wedge_{i\in I}\mathcal{Q}_i} \subset \wedge_{i\in I}\tau_{\mathcal{Q}_i}$.

d) If $\wedge_{i\in I}\mathcal{Q}_i = \cap_{i\in I}\mathcal{Q}_i$, then $\tau_{\mathcal{Q}_\wedge} = \wedge_{i\in I}\tau_{\mathcal{Q}_i}$.

e) If $I = \{1,...,n\}$ the family $\mathcal{B} := \{Q_1 \circ ... \circ Q_n : Q_i \in \mathcal{Q}_i\}$ is a base of some local quasi-uniformity $\mathcal{Q}$ then $\tau_{\mathcal{Q}} = \tau_{\wedge_{i=1}^n \mathcal{Q}_i}$.

**Proof:**

a) and b) are easy to check.

$c)$ Follows from $b)$ because $\wedge_{i\in I}\mathcal{Q}_i \subset \cap_{i\in I}\mathcal{Q}_i$.

d) Follows from $b)$.

e) Evidently, for each $i \in I$ we have $\mathcal{Q} \subset \mathcal{Q}_i$, and $\mathcal{Q} \subset \bigcap_{i=1}^{n}\mathcal{Q}_i$. By the definition of the greatest lower bound we have $\mathcal{Q} \subset \wedge_{i=1}^n \mathcal{Q}_i$, hence $\tau_{\mathcal{Q}} \subset \tau_{\wedge_{i=1}^n \mathcal{Q}_i}$.

Let us show that $\tau_{\mathcal{Q}} \supset \tau_{\wedge_{i=1}^n \mathcal{Q}_i}$. Take $G \in \tau_{\wedge_{i=1}^n \mathcal{Q}_i}$ and $x \in G$, there is a $U$ such that $U[x] \subset G$. For each $U \in \wedge_{i=1}^n \mathcal{Q}_i$ we have $V \in \wedge_{i=1}^n \mathcal{Q}_i$, such that

$$V \circ ... \circ V[x] \subset U[x].$$

Since for each $i = 1,...,n$, $V \in \mathcal{Q}_i$, the set $V \circ ... \circ V$ belongs to $\mathcal{B}$, which is a base of $\mathcal{Q}$. Then there is an entourage $V \circ ... \circ V$ such that each $x \in X$ we have $V \circ ... \circ V[x] \subset U[x] \subset G$, consequently $G \in \tau_{\mathcal{Q}}$. ♦

**Remark 4.7**

1. If $Q$ is a quasi-uniformity, the 4.6(e) is a particular case of 3.16.
2. If $(X, Q^{\top})$ is local quasi-uniform then $Q$ is weakly locally symmetric at $x \in X$ if and only if $\mathcal{N}_{\tau_Q}(x) \subset \mathcal{N}_{\tau_{Q^{\top}}}(x)$ (it's direct consequence of 3.12).

With the following example we can see that in general the inclusions of the 3.15(b) and 4.6(c) can be strict.

**Example 4.8**

Let $X$ be an infinite set, and $\tau$ be $T_2$ topology in $X$ such that any $\tau$-continuous function $f : X \to [0,1]$ is constant and $Q := Q^{Per}(\tau)$ (see 3.5). Then:

a) $\tau_{Q_\wedge} = \{\emptyset, X\}$.

To prove this, suppose that $\tau_{Q_\wedge} \neq \{\emptyset, X\}$. Hence $\tau_{Q_\wedge}$ is a completely regular topology, which is not indiscrete. This implies that there is a non-constant $\tau_{Q_\wedge}$-continuous $f : X \to [0,1]$. Since $\tau_{Q_\wedge} \subset \tau$ then $f : X \to [0,1]$ is $\tau$-continuous as well, but this contradicts our choice of $\tau$. Consequently, $\tau_{Q_\wedge}$ is the indiscrete topology.

b) $\tau_{Q_\wedge} \neq \tau_Q \cap \tau_{Q^{\top}}$.

Follows from the first statements because $\tau_Q \cap \tau_{Q^{\top}}$ is a $T_1$-topology.

c) $Q_\wedge \neq Q \cap Q^{\top}$.

It is a immediate consequence of b).

A topological space $(X, \tau)$ is called local quasi-uniformizable**,** (resp. quasi-uniformizable, local uniformizable, uniformizable) if there is a local quasi-uniformity (resp. quasi-uniformity, local uniformity, uniformity) $Q$ such that the topology induced by $Q$ is $\tau$, i.e. $\tau_Q = \tau$. When this uniform type structure $Q$ is unique we say that $(X, \tau)$ is uniquely local quasi-uniformizable, (resp. quasi-uniformizable, local uniformizable, uniformizable).

In this setting, there are two classic results that we have to mention. The first is due to Weil, and it shows that a "topological space is uniformizable if and only if it is completely regular".
The second theorem assures that any "topological space is quasi-uniformizable" and it was proved by Krishnan(1955). Later Császér(1960), showed this result, but subsequently Pervin gave a more direct and simpler proof. Pervin proved that for any topological space $(X,\tau)$, the topology generated by the quasi-uniformity $Q^{Per}(\tau)$ (see 3.5) is $\tau$.

Although the local quasi-uniformity formally has weaker properties than the quasi-uniformity, however to construct the local quasi-uniformity compatible with a given topology seems not to be easier, than to build a quasi-uniformity with the same property.

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